EQUATIONS OF NON-VERTICAL PLANES in 3-SPACE.

EVERY PLANE $\mathbb{P}$ in $\mathbb{R}^3$ has an equation of the form:

$\mathbb{P}$: $ax + by + cz = d$ for some constants a, b, c, d.

Such an equation is called a <u>Linear Equation</u> of $\mathbb{P}$.

ALSO, the plane $\mathbb{P}$ is NON-VERTICAL $\Leftrightarrow$ $c \neq 0$.

In this paper, we show <u>how to find a linear equation of a given non-vertical plane</u> $\mathbb{P}$ when all we know about plane $\mathbb{P}$ is that <u>it passes through three given points, (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2).</u>

<u>FACT</u>: If we know the plane $\mathbb{P}$ is Non-Vertical and passes through the given point (x_0, y_0, z_0), Then, an equation of $\mathbb{P}$ is

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

for particular numbers A and B.

We will know an equation for $\mathbb{P}$ once we have determined what numbers A and B are.

When we know two other points (x_1, y_1, z_1) and (x_2, y_2, z_2) on plane $\mathbb{P}$, we can use those two points to determine A and B. Since those two points are also on plane $\mathbb{P}$, Their coordinates will satisfy the equation

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

when $(x, y, z) = (x_1, y_1, z_1)$ and

when $(x, y, z) = (x_2, y_2, z_2)$.

We substitute these coordinates into the equation to obtain two linear equations in the two unknowns A and B. We then solve for A and B.

Then, finally, we convert the equation

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

into its <u>linear equation</u> form $ax + by + cz = d$.

<u>EXAMPLE 1</u>: It is given that plane $\mathbb{P}$ is non-vertical and passes through the points

$(3, 5, 1)$, $(2, 3, 0)$, $(1, 4, 1)$.

Determine a linear equation of Plane $\mathbb{P}$.

<u>Solution</u>: It is arbitrary which of these points is to be called (x_0, y_0, z_0) — (It doesn't matter which it is) — so, let's set $(x_0, y_0, z_0) = (3, 5, 1)$.

(EXAMPLE 1 CONTINUED)

So, using $(x_0, y_0, z_0) = (3, 5, 1)$, the equation $z - z_0 = A(x - x_0) + B(y - y_0)$ becomes:

$$z - 1 = A(x-3) + B(y-5). \qquad (*)$$

Substituting in the points $(x_1, y_1, z_1) = (2, 3, 0)$ and $(x_2, y_2, z_2) = (1, 4, 1)$ into $(*)$, we get equations ① and ② below:

① $0 - 1 = A(2-3) + B(3-5)$ ①

$\Rightarrow -1 = -A - 2B \Rightarrow \underline{1 = A + 2B}$

and

② $1 - 1 = A(1-3) + B(4-5)$ ②

$\Rightarrow 0 = -2A - B \Rightarrow B = -2A$

According to the final form of equation ②, we can substitute $-2A$ in for B in Equation ①, which gives us:

$1 = A + 2(-2A) \Rightarrow 1 = A - 4A \Rightarrow -3A = 1$

$\Rightarrow A = -\frac{1}{3}$

Since $B = -2A$, $B = +\frac{2}{3}$

An equation for $\mathbb{P}$, then, is $z - 1 = -\frac{1}{3}(x-3) + \frac{2}{3}(y-5)$.

multiplying through by 3: $3z - 3 = -(x-3) + 2(y-5)$

$\Rightarrow 3z - 3 = -x + 3 + 2y - 10 \Rightarrow 3z - 3 = -x + 2y - 7$

$\Rightarrow \underline{x - 2y + 3z = -4}$.

$\boxed{\text{A linear EQUATION of } \mathbb{P} \text{ is } x - 2y + 3z = -4}$

EXAMPLE 2 : It is given that plane $\mathbb{P}$ is non-vertical and passes through the points $(2,-1,4)$, $(1,1,-1)$, $(-4,1,1)$.

Determine a linear equation for $\mathbb{P}$.

Sol'n: Let $(x_0, y_0, z_0) = (2,-1,4)$.

Then, the equation $z - z_0 = A(x-x_0) + B(y-y_0)$

becomes $z-4 = A(x-2) + B(y+1)$ $(*)$

Substituting the points $(x_1,y_1,z_1) = (1,1,-1)$ and $(x_2,y_2,z_2) = (-4,1,1)$ into $(*)$,

we obtain equations ① and ②:

① $-1-4 = A(1-2) + B(1+1) \Rightarrow -5 = -A+2B$
$\Rightarrow 2B = A-5$ ①

② $1-4 = A(-4-2) + B(1+1) \Rightarrow -3 = -6A+2B$
$\Rightarrow 2B = -3+6A$ ②

Setting the two expressions for $2B$ equal to each other, we have $A-5 = -3+6A \Rightarrow -2 = 5A \Rightarrow A = \frac{-2}{5}$

so, $2B = -\frac{2}{5} - 5 = \frac{-27}{5} \Rightarrow B = \frac{-27}{10}$

One equation for $\mathbb{P}$ is $z-4 = \left(\frac{-2}{5}\right)(x-2) + \left(-\frac{27}{10}\right)(y+1)$.

multiplying through by 10 gives $10z - 40 = -4(x-2) - 27(y+1)$

$(21 = 8 - 27 + 40)$

This simplifies to

$\boxed{4x + 27y + 10z = 21}$